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J. Phys. A: Math. Gen. 21 (1988) 1157-1165. Printed in the UK

Classical billiards in a rotating boundary

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Received 22 October 1987

Abstract. As a specific example of billiards in a moving boundary, we discuss the classical motion on a circular table rotating uniformly about a point on its edge. Computer evidence for periodic and chaotic motion is displayed and we expose the origin of the chaotic motion as due to the effects of curvature of trajectories in the rotating frame.

1. Introduction

There have been many investigations in recent years of the classical billiard problem (elastic collisions with a fixed boundary of a freely moving point particle) in various geometries with a view to understanding the phenomena giving rise to chaotic motion. Variations have included polygonal boundaries with discontinuities in tangent [1] boundaries with discontinuities in curvature (stadium of Bunimovich) [2] boundaries with a cusp [3] and inclusion of magnetic fields [4] to give a time-reversal asymmetric system [5]. The essential feature of all these systems is that, when projected from a given point on the boundary, the angle of return of the billiard after the first bounce is not a continuous function of the angle of projection, for a given energy, though the mechanism whereby this is achieved varies from case to case. All of these systems for the study of two-dimensional chaotic motion have the advantage over the mathematically simpler Henon attractor, which is just a quadratic map, of arising from idealised physical situations. The system which we present here is another variant, which combines mathematical simplicity (the boundary is C^{∞}) with physical realisation, and incorporates time-reversal non-invariance. It is the motion of a billiard on a smooth circular table which is rotating with uniform angular velocity about a point, taken for simplicity on the edge of the circle, about an axis perpendicular to the plane of the table. It is interesting to recall the work of Birkhoff [6]: 'Any Lagrangian system with two degrees of freedom is isomorphic with the motion of a particle on a smooth surface rotating uniformly about a fixed axis and carrying a conservative field of force with it'. Indeed the system under consideration, in the frame of the table, is equivalent to the motion of an electron in a constant vertical magnetic field and in a linear electric field.

2. The equations of motion

It is convenient to work in a frame rotating with the table, and to parametrise the x, y coordinates of the billiard in that frame by the complex variable $z = x + iy$. Taking the

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billiard of unit mass and scaling z, t so that the circle has unit radius, and the angular velocity is also unity, then the equation of motion, using the standard transformation to a rotating frame, is just

$$\ddot{z}+2i\dot{z}-z=0 \tag{1}$$

with general solution

$$z=(a+bt)\,e^{-it} \tag{2}$$

where a and b are complex constants.

We may take the equation of the boundary to be

$$|z-1|=1 \tag{3}$$

so that the axis of rotation lies at the point $(0,0)$ on the circumference of the circle. Equation (2) is just the transformation to the rotating frame of rectilinear motion. The reason for working in the rotating frame is that the conditions at the bounce are easier to formulate: if $\dot{z}'$ denotes the velocity of the particle after a bounce then

$$\dot{z}'=-\dot{\bar{z}}(z-1)^2 \tag{4}$$

at the boundary. Equation (4) is just an algebraic way of stating that components of velocity tangential to the circle are unchanged at a collision with the boundary, whereas perpendicular components are reflected.

The Lagrangian from which equation (1) may be derived is

$$|\dot{z}|^2+|z|^2+i(\dot{\bar{z}}z-\dot{z}\bar{z}) \tag{5}$$

and the motion may be interpreted as that of a charged particle in a Lorentz force

$$F=z-2i\dot{z}$$

i.e. that arising from a constant magnetic field of magnitude 2 perpendicular to the plane, and a linear electric field directed radially outwards. The associated Hamiltonian density is

$$H=p\bar{p}+i(\bar{q}p-q\bar{p}) \tag{6}$$

where

$$q=z \qquad p=\dot{z}+iz=\dot{q}+iq. \tag{7}$$

Note that the Hamiltonian is a Hermitian, rather than symmetric quadratic form as a consequence of the lack of time-reversal invariance (we are supposing that in the reversed motion the angular velocity of rotation remains the same).

This system gives rise to three related constants of the motion: an energy

$$H=|\dot{z}|^2-|z|^2=|\dot{q}|^2-|q|^2 \tag{8}$$

a momentum

$$P=|p| \tag{9}$$

and an angular momentum

$$A=i(\bar{q}p-q\bar{p})=i(\bar{z}\dot{z}-z\dot{\bar{z}})+2|z|^2 \tag{10}$$

$$H=P^2+A. \tag{11}$$

The energy remains conserved throughout: the momentum and angular momentum change at each collision

$$\delta A = \mathrm{i}(\dot{z}' - \dot{z} + \dot{\bar{z}} - \dot{\bar{z}}'). \tag{12}$$

When the particle is on the boundary we may parametrise

$$z = 1 + \mathrm{e}^{\mathrm{i}\theta} = 2\,\mathrm{e}^{\mathrm{i}\theta/2}\cos(\theta/2) \qquad |\dot{z}| = v.$$

Then from (8)

$$v^2 - 4\cos^2(\theta/2) = 2K - 4 = \text{constant} \tag{13}$$

where K is the kinetic energy of the particle at $\theta = 0$.

3. Fixed points and periodic orbits

The fixed points and periodic orbits (n-cycles) of a Hamiltonian system reveal much about the dynamics of the system, and in particular the study of the periodic orbits is relevant to the quantisation of the system, and the stability is relevant to the nature of the eigenvalues. In one-dimensional maps the bifurcation of n-cycles into larger n-cycles occurs as an infinite sequence of period doublings as a parameter is varied. Here we have a two-dimensional conservative system, with a correspondingly richer set of possibilities. The general nature of such maps has been discussed at great length by Greene *et al* [7]. In this problem the specific two-dimensional map is that from (θ_0, ψ_0) to (θ_1, ψ_1) where θ_0 is the initial angle on the circle, ψ_0 the angle of the initial velocity with the tangent at θ_0, and θ_1, ψ_1 the corresponding angles at the next intersection of the trajectory with the boundary (figure 1). If this map is denoted by

$$f\colon (\theta_0, \psi_0) \to (\theta_1, \psi_1)$$

then for an n-cycle

$$f^n(\theta, \psi) = (\theta, \psi).$$

Fixed points of f occur only when the energy of the system is sufficiently low for closure of a trajectory: they also necessitate a zero change in the angular momentum (12), i.e.

$$(\dot{z}\,\mathrm{e}^{\mathrm{i}\theta} + \dot{\bar{z}}\,\mathrm{e}^{-\mathrm{i}\theta})\sin\theta = 0 \tag{14}$$

i.e. $\theta = 0$, π or $\dot{z} = \mathrm{i}v\,\mathrm{e}^{-\mathrm{i}\theta}$, the velocity at collision is tangential.

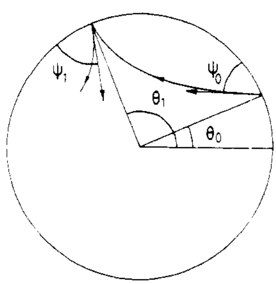

Figure 1. A typical trajectory and reflection in the rotating frame.

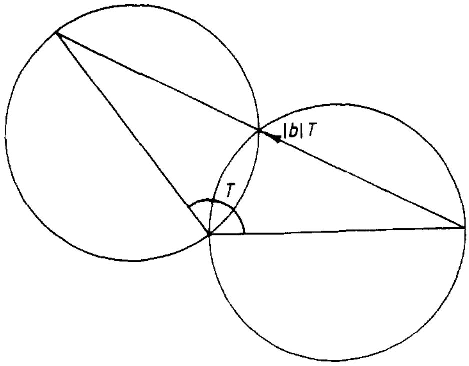

Figure 2. A 1-cycle as seen in the laboratory frame.

In fact the only viable possibility is $\theta=0$, which may be seen by a symmetry argument, or else algebraically as follows. The condition for the trajectory to cross at time T on the circle at angle θ is, from (2),

$$2\,e^{i\theta/2}\cos(\theta/2)=[2\,e^{i\theta/2}\cos(\theta/2)+bT]\,e^{-iT} \tag{15}$$

$$\text{Initial velocity}=-2i\,e^{i\theta/2}\cos(\theta/2)+b=\text{final velocity}=-2i\,e^{i\theta/2}\cos(\theta/2)+b\,e^{-iT} \tag{16}$$

since the velocity is tangential for $\theta\neq 0$, π. These equations have no non-trivial solution unless $\theta=0$, when (16) is replaced by

$$b=-\bar{b}\,e^{-iT} \tag{17}$$

obtained by equating the initial velocity at $\theta=0$ to the velocity after the first bounce. Then (15) and (17) are compatible and give

$$|b|T=4\sin(T/2). \tag{18}$$

This equation can also be derived from simple geometrical considerations in the laboratory frame.

$|b|$ is the speed of the particle in the laboratory frame. From figure 2 the circle will have turned through an angle T in the time the particle takes to hit the boundary, after a trajectory of length $|b|T$. Simple geometrical considerations give the condition (18) for the existence of a 1-cycle provided that it is energetically possible. It is also clear from figure 2 that a billiard projected from the opposite end of the diameter towards the origin with any velocity whatever will perform a 2-cycle: in the rotating frame this translates into a statement about the minimal kinetic energy: the angle ψ of projection is given by

$$v\cos\psi=(2K)^{1/2}\cos\psi=2 \tag{19}$$

which requires $K\geqslant 2$ for a solution, i.e. for the existence of a 2-cycle.

4. Study of the first iteration of the map

Many of the effects of the bounce maps may be understood and traced back to a property of one iteration of the map f: $(\theta_0,\psi_0)\to(\theta_1,\psi_1)$ where the angles are those in figure 1.

For example, consider a set of trajectories starting from the same point on the edge, but at any angle ψ_0 between 0 and π. A typical set of trajectories is shown in figure 3.

Some of the trajectories have 'looped', and split off from the main group; there is a discontinuity in the angle θ_1 which occurs at the trajectory which just glances the edge of the circle. There are two of these in this case.

Here we see the origin of the chaotic motion from the discontinuities in the map f: the mechanism which produces them here is the curvature of the trajectories, a feature previously studied only in the case of magnetic fields [3]. However, there the picture is confused by the presence of other effects due to variations and discontinuities in the curvature of the boundaries. These trajectories are important, as they determine whether the system exhibits chaos, because they mark a point where nearby trajectories diverge. For many iterations this would extend to many points, and there would be 'mixing'. These special trajectories can only occur if the curvature of the trajectory is greater than the curvature of the circle. We can plot the mapping of figure 3 onto a phase portrait as in figure 4.

The most comprehensive way of displaying the mapping is to consider not only trajectories with the same initial θ, but also to vary θ_0 as well as ψ_0. This results in figure 4 which displays the mapping as a two-dimensional curvilinear coordinate

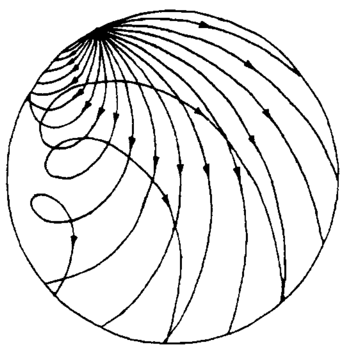

Figure 3. A set of trajectories originating from one point with the same energy but different initial directions, illustrating the discontinuity in the position of the next bounce.

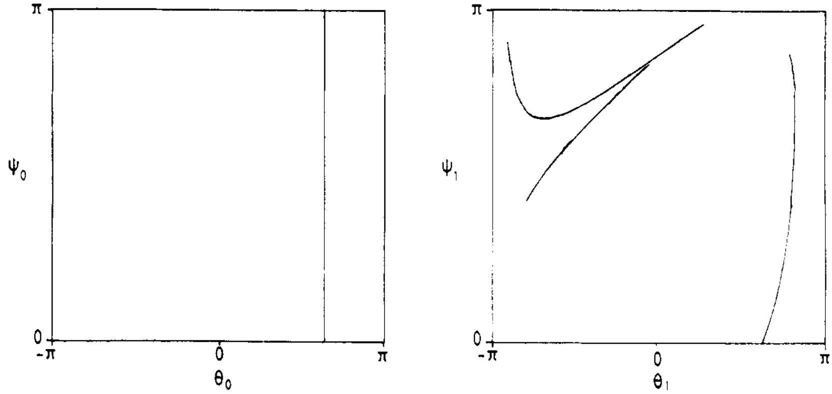

Figure 4. Phase portraits of the mapping f: $(\theta_0, \psi_0) \rightarrow (\theta_1, \psi_1)$.

transformation. The more complex the mapping, the more likely it is to show chaotic behaviour.

5. Perturbative methods of approximation

We must look at approximate solutions to the problem for special initial values in order to study the system further. Suppose that the energy of the particle is sufficiently large that the circle does not rotate very far before the particle hits the boundary. Then we can approximate T as small, so

$$z(T) \simeq (a+bT)(1-\mathrm{i}T-\tfrac{1}{2}T^2)$$

which gives

$$T = \frac{-2\,\mathrm{Re}(V\,\mathrm{e}^{-\mathrm{i}\theta})}{|V|^2 + 2\,\mathrm{Im}(U\,\mathrm{e}^{-\mathrm{i}\theta})}$$

where V is the initial velocity of the particle in the rotating frame at $t=0$ and $U = V + \frac{1}{2}\mathrm{i}(1+\mathrm{e}^{\mathrm{i}\theta})$.

When the particle is projected almost tangential to the edge of the circle

$$T \simeq \frac{2v\varepsilon}{\cos\theta + (1+v)^2}$$

where $V = \mathrm{i}v\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{e}^{\mathrm{i}\varepsilon}$. Further calculations show that

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\theta} \simeq \frac{\Delta\varepsilon}{\Delta\theta} = 0$$

so ε is a constant, but the angle θ changes by $\Delta\theta$ each iteration where

$$\Delta\theta = \frac{2\varepsilon v^2}{\cos\theta + (1+v)^2}$$

$$v^2 = 2(\cos\theta + K - 1).$$

To see the change in ε we must expand to powers of ε^2 which requires expanding $\mathrm{e}^{-\mathrm{i}T}$ to the T^3 term. This requires a long calculation which was aided by a REDUCE program. The end result was the answer

$$\varepsilon = Av^{-1}(3v^2+4v+4-2K)^{1/3}$$

where A is a constant determined by the initial value of $\varepsilon = \psi_0$.

6. Symmetries of the periodic orbits

The equation of motion is invariant under the transformation $t \to -t$, $z \to \bar{z}$, $\dot{z} \to \dot{\bar{z}}$. This means that, for a periodic orbit, if z is a point of intersection with the boundary so will $\bar{z}$, and thus the orbits are symmetric about the line $\theta = 0$. In particular this means that the 1-cycles must start from $\theta = 0$ (trajectories from $\theta = \pi$ must hit the boundary elsewhere). However, a study of the iterative mapping in figure 6 shows an asymmetry in the chaotic region: only for the integrable region can the time be reversed and tne symmetry about $\theta = 0$ deduced.

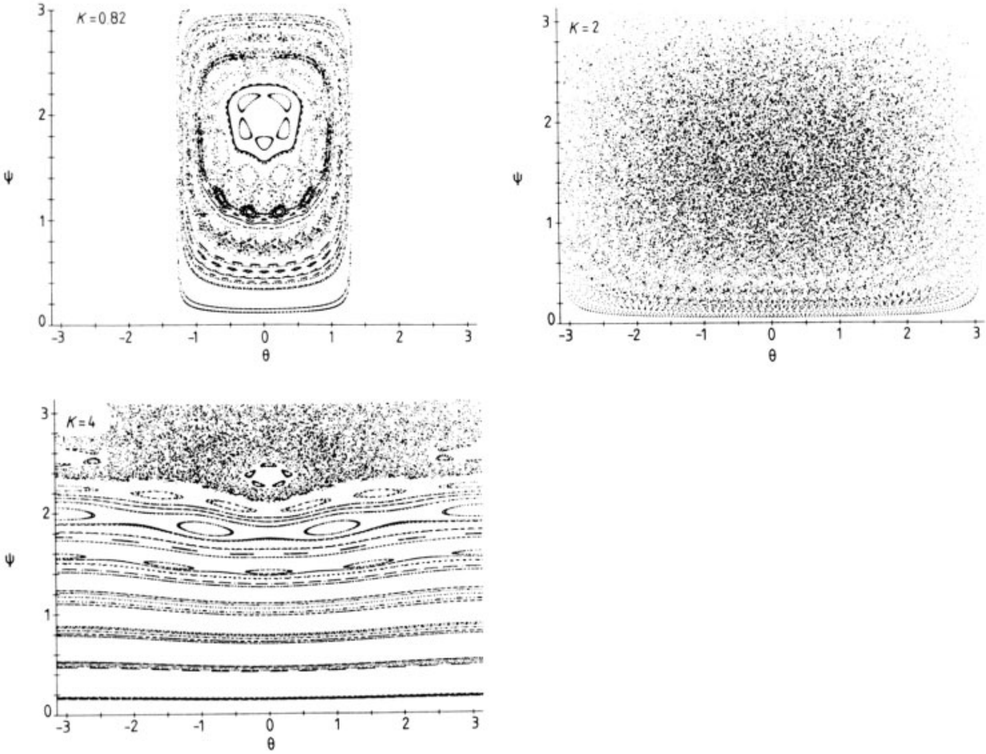

Figure 5. Phase portraits of the whole mapping region, as explained in § 6.

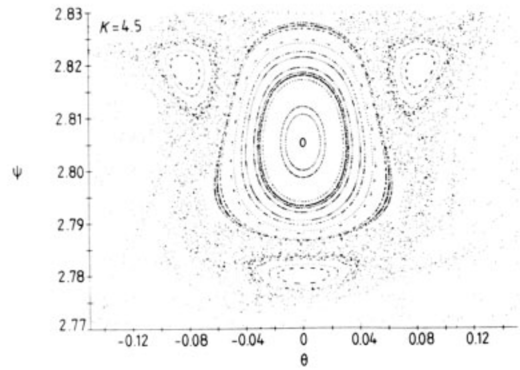

Figure 6. A small region expanded from the top centre of the phase portrait at $K = 4.5$, as explained in § 6.

The iteratives (θ_i, ψ_i) for several paths can be followed for a few hundred iterations. The resulting graph of ψ against θ is a cross section through phase space.

Figure 5 shows phase portraits of the whole mapping region $\psi = 0$ to π and $\theta = -\pi$ to π for three energies: at $K = 0.82$ showing mostly undestroyed tori; at the critical energy $K = 2$ showing no tori; and at $K = 4$ showing mostly undestroyed tori for $\psi \lesssim 3\pi/4$ and destroyed tori for $\psi \gtrsim 3\pi/4$.

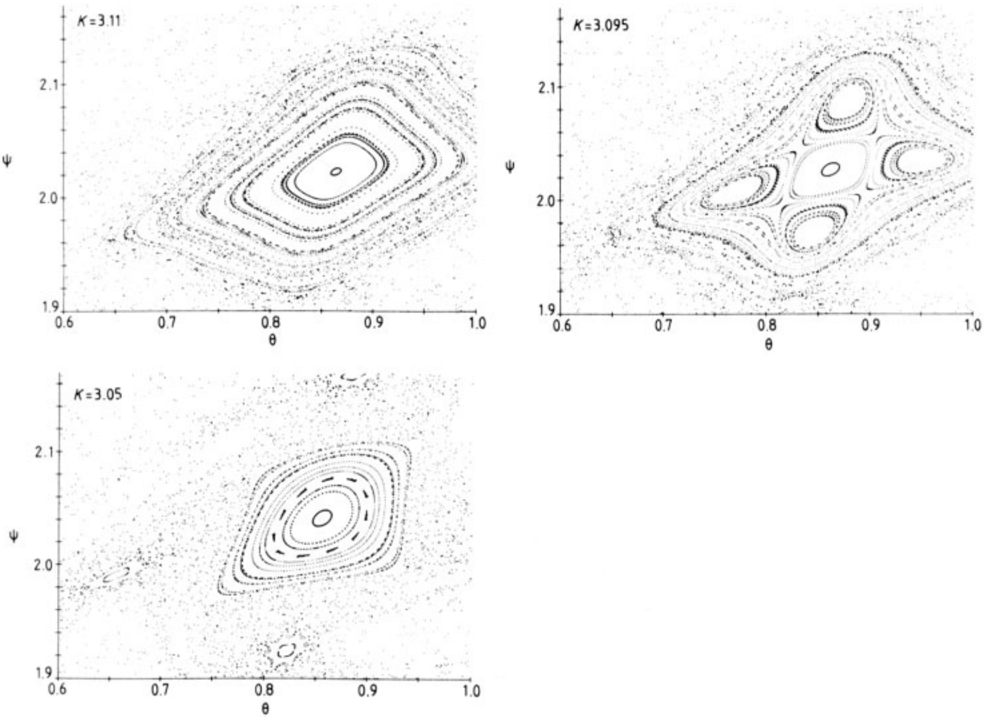

Figure 7. One point of a 3-cycle and the birth of a 12-cycle, as explained in § 6.

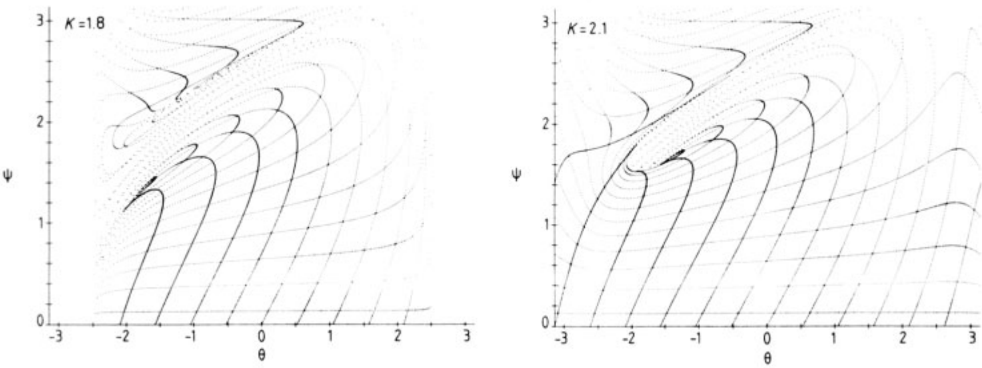

Figure 8. The mapping at F as explained in §§ 4 and 6.

Figure 6 shows a small region expanded from the top centre of the phase portrait at $K = 4.5$. The remaining undestroyed tori have reflectional symmetry about $\theta = 0$, but where tori have been destroyed near hyperbolic points, this symmetry is broken.

Figure 7 shows one point of a 3-cycle, and the birth of a 12-cycle: at $K = 3.11$, before birth; at $K = 3.095$, after birth; and at $K = 3.05$, destruction of the 12-cycle tori.

Figure 8 shows the mapping of f as explained in § 4: at energies $K = 1.8$ and 2.1.

7. Conclusion

An initial study of a very simple dynamical system, the rotating circular billiard table, demonstrates the expected features of chaotic systems: isolated n-cycles, bifurcations, chaotic regions, etc. The system, lacking time-reversal invariance, may have some consequence for the understanding of the quantum problem. The spectrum of the Hamiltonian (6) in the circular boundary is a challenging question.

Acknowledgment

This research was supported by a studentship from the SERC.

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